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Option Pricing: A Simplified Approach†

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[1978 winner of the Pomeranze Prize of the Chicago Board Options Exchange]

[reprinted in *Dynamic Hedging: A Guide to Portfolio Insurance*, edited by Don Luskin (John Wiley and Sons 1988)] : 책에 실림.

[reprinted in *The Handbook of Financial Engineering*, edited by Cliff Smith and Charles Smithson (Harper and Row 1990)]

[reprinted in *Readings in Futures Markets* published by the Chicago Board of Trade, Vol. VI (1991)]

[reprinted in *Vasicek and Beyond: Approaches to Building and Applying Interest Rate Models*, edited by Risk Publications, Alan Brace (1996)]

[reprinted in *The Debt Market*, edited by Stephen Ross and Franco Modigliani (Edward Lear Publishing 2000)]

[reprinted in *The International Library of Critical Writings in Financial Economics: Options Markets* edited by G.M. Constantinides and A..G. Malliaris (Edward Lear Publishing 2000)]

Abstract

This paper presents a simple discrete-time model for valuing options. The fundamental economic principles of option pricing by arbitrage methods are particularly clear in this setting. Its development requires only elementary mathematics, yet it contains as a special limiting case the celebrated Black-Scholes model, which has previously been derived only by much more difficult methods. The basic model readily lends itself to generalization in many ways. Moreover, by its very construction, it gives rise to a simple and efficient numerical procedure for

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† Our best thanks go to William Sharpe, who first suggested to us the advantages of the discrete-time approach to option pricing developed here. We are also grateful to our students over the past several years. Their favorable reactions to this way of presenting things encouraged us to write this article. We have received support from the National Science Foundation under Grants Nos. SOC-77-18087 and SOC-77-22301.

- 이 사람들은 미방이 아닌 아주 쉬운 수준의 수학만으로 같은 모델을 만들 수 있음을 증명하고 있다.
  American option도 고려하고 있음. 이는 만기 이전에도 행사할 수 있다. 보통은 American option에서 미분방정식을 풀 수 없는데, 이 사람들은 Exact한 solution은 주지 못해도 numerical한 procedure는 줄 수 있다는 것이다. (즉 수치해석적 기법으로 가능하다는 의미이다)
  다만 American option에 대한 부분은 빼고 European option 부분만 다룰 예정이다.

: William Sharpe란 사람이 아이디어를 주었다. 그 아이디어 가지고 논문을 낸 것이다! 그리고 그거 하면서 학생들에게 가르쳐 본 것이다. 대학원 수준의 책이라면 대부분 처음 start는 강의 노트이다. (Hull 책도 처음 시작은 강의노트이다) 그거 가지고 가르치다 보면 학생들이 어느 부분을 이해하고 이해하지 못하고를 알 수 있다. 그런걸 계속 반영하다 보면 책이 되는 것이다.

- NSF: 우리 식으로 치면 연구 기금만을 학자들에게 대 주는 곳이다. 미국 교수들은 급여를 받는데 특이한 점은 가르칠 때만 급여를 받는다. 그래서 나머지 기간에는 NSF에서 이런이런 연구를 하겠다고 하고 먹고 사는 것이다. 이런 식으로 Research fund를 받는 것이다.
- 미국에 유학가면 RA/TA를 하는데, RA는 많은 경우에 학생에게 직접 월급처럼 돈을 주는 것이다. 그런데 그 월급의 Source가 NSF이다.

1. Introduction

An option is a security that gives its owner the right to trade in a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date. The act of making this transaction is referred to as exercising the option. The fixed price is termed the strike price, and the given date, the expiration date. A call option gives the right to buy the shares; a put option gives the right to sell the shares.

Options have been traded for centuries, but they remained relatively obscure financial instruments until the introduction of a listed options exchange in 1973. Since then, options trading has enjoyed an expansion unprecedented in American securities markets.

- 공식적인 장내 시장을 말함.

Option pricing theory has a long and illustrious history, but it also underwent a revolutionary change in 1973. At that time, Fischer Black and Myron Scholes presented the first completely satisfactory equilibrium option pricing model. In the same year, Robert Merton extended their model in several important ways. These path-breaking articles have formed the basis for many subsequent academic studies.

- 우리나라 옵션 시장만큼 활발한 시장이 없다! (어떻게 보면 베팅이라서 그렇다)
- 노벨상의 기준 - Path-breaking한 article을 쓰고, 그걸로 인해 많은 새로운 연구를 불러 일으켜야 한다. (따라서 노벨상은 항상 늦게 받을 수 밖에 없다.)

As these studies have shown, option pricing theory is relevant to almost every area of finance. For example, virtually all corporate securities can be interpreted as portfolios of puts and calls on the assets of the firm.[1] Indeed, the theory applies to a very general class of economic problems — the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event.

- **부도(Default)란 무엇인가?**

For example, virtually all corporate securities can be interpreted as portfolios of puts and calls on the assets of the firm.[1] Indeed, the theory applies to a very general class of economic problems — the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event.

Unfortunately, the mathematical tools employed in the Black-Scholes and Merton articles are quite advanced and have tended to obscure the underlying economics. However, thanks to a suggestion by William Sharpe, it is possible to derive the same results using only elementary mathematics.[2]

In this article we will present a simple discrete-time option pricing formula. The fundamental economic principles of option valuation by arbitrage methods are particularly clear in this setting. Sections 2 and 3 illustrate and develop this model for a call option on a stock that pays no dividends. Section 4 shows exactly how the model can be used to lock in pure arbitrage profits if the market price of an option differs from the value given by the model. In section 5, we will show that our approach includes the Black-Scholes model as a special limiting case. By taking the limits in a different way, we will also obtain the Cox-Ross (1975) jump process model as another special case.

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[1] To take an elementary case, consider a firm with a single liability of a homogeneous class of pure discount bonds. The stockholders then have a "call" on the assets of the firm which they can choose to exercise at the maturity date of the debt by paying its principal to the bondholders. In turn, the bonds can be interpreted as a portfolio containing a default-free loan with the same face value as the bonds and a short position in a put on the assets of the firm.

[2] Sharpe (1978) has partially developed this approach to option pricing in his excellent new book, *Investments*. Rendleman and Bartter (1978) have recently independently discovered a similar formulation of the option pricing problem.

2

- **BIS 비율**
  또한 사람들은 이걸 가지고 무엇을 구하냐? 은행의 BIS 비율이라는 것을 계산한다. 가장 최근의 BIS 비율은 이야기는 아마 외환은행 사례일 것이다. 이 BIS 비율을 선정하는데 볼튼의 생각을 반영한 것이다. 은행은 기본적으로 대출을 해 줘야 먹고사는 곳이다. 은행이 100억원의 대출을 해 줬다고 해 보자. 은행에다 예금을 맡기면 기업들에게 대출을 해준다. 대출해주고 나서도 자기 자본을 가지고 있어야 하는데, 그 비율이 8%가 되어야 하는 것이 BIS 비율이다. 그런데 여기에서 문제가 발생할 수 있다. 만약 우리가 은행을 경영한다고 해 보자. BIS 비율의 비중이 8%이므로 8%만 은행에 쌓아두고 있으면 된다. 그런데 돈을 더 벌고 싶다면 어떻게 해야 하나? 수익성이 높은 곳에다 대출을 해 주면 된다. 즉 이자를 많이 받는 곳에다 대출해 주게 된다. 그런데 문제는 그런 식의 이자를 많이 받을 수 있는 곳은 대개 Risk가 높은 곳이란 점이다. (삼성보다는 중소기업의 경우 이자가 더 높기 때문이다) 따라서 이런 식으로 Fixed rule이 있으면 위험이 생긴다. 즉 이 BIS의 목적은 금융기관들이 안 망하도록 하는 룰인데, 이걸 잘못 악용하면 금융기관들은 오히려 더 위험해지는 것이다. (자꾸 대출금 회수를 못하면 은행이 망하기 때문이다)

- **Basel II**
  BIS = Bank for International Settlements라고 하고, 우리말로는 국제 결제 은행이라고 한다. BIS는 기본적으로 국제 금융기구이다. 각 국가의 중요한 기

없다.)

- **부도(Default)란 무엇인가?**
- 이를 정의한 것이 Merton이다. 기업의 재무제표를 보면 왼쪽 Asset, 오른쪽 Liability/Equity가 있다. 그런데 L은 다른 사람으로부터 돈을 빌린 것이다. (혹은 채권을 발행할 수도 있다) 또한 E는 자신이 주식을 발행하는 것이다. 따라서 L에 대해서는 정해진 이자를 주어야 하고 E에 대해서는 이자를 주는 것이 아니다. (엄밀히 말해 E는 배당만을 갖고 끝나는 것이다.)
- Merton은 회사를 대단히 simple하게 생각했다. 만기시점 T가 되어 회사를 청산한다고 해 보자. L을 K라 하는 특정 금액을 만기 시점이 되기 전에는 아무 것도 안 주다가 만기 시점에 K를 주는 채권이라고 생각하여 보자. 그런 식의 채권을 zero-coupon bond라고 한다. (즉 이자가 없는 채권이다) 그럼 이런 채권을 왜 사나? 지금으로부터 3년 후에 100이라고 하는 돈을 받을 수 있다? 그럼 현재 시점에서 이의 가격은 100보다는 쌀 것이다. (이자를 안 주기 때문)
- E는 주식을 발행해서 돈을 모아서, 여하튼 L+E로부터 모아진 돈으로 공장을 지어 생산을 하면 전체적인 A가 높아지게 되는 것이다.
- 그럼 부도란 무엇인가? L과 E를 가지고 있는 사람들이 누가 우선순위가 있나? 금액이 적으면 L을 충분히 갚을 만한 자산이 안 되면? 그럼 L을 가진 사람이 전부를 가지게 된다. 따라서 이를 수식으로 표현하면,
  - 만기 시점의 경우

$$A_T > K$$

$$A_T < K$$

	Bond holder	Equity holder
$A_T>K$	K	A_T - K
$A_T<K$	**A_T**	**0**

이 중 아래 진하게 표시된 것이 바로 부도(default)로 정의될 수 있다.

이 중 K는 고정되어 있는 숫자이다. 그리고 A_T는 랜덤하게 결정된다. 따라서 **Max(A_T-K, 0)**과 같다. 결국 주식을 하나 가지고 있다는 것은 그 기업에 대한 콜옵션을 가지고 있다고 생각해 볼 수도 있는 것이다.

그럼 왼쪽의 경우는 어떻게 되나? **Min(K, A_T)**가 될 것이다. 왜? A_T>K일 때 K를 가져가고 A_T<K일 때 A_T를 가져가기 때문이다. 이게 옵션인가? 이건 옵션이 아니다. 그런데 이걸 아래와 같이 바꿀 수 있다.
Min(K, A_T) = K - Max(K-A_T, 0)

그런데 Max(K-A_T,0)이걸 보면 풋옵션과 같다. 그런데 앞에 -가 붙었다? 즉 풋

은행이라고 한다. BIS는 기준적으로 국제 금융기구이다. 각 국가의 주요한 기관들이 돈을 내서 만든 곳이다. 이들은 권한은 없다. 그런데 이 Suggestion이 있는데, 그게 거의 Rule이 된다. (왜냐하면 세계 대부분의 은행이 BIS를 따르게 되기 때문이다)
이거 안 따르는 대표적인 은행의 대표적인 예는 중국 은행들이다. (그래서 국제 금융 시장에서 놀 수 있는 은행들이 아니다.)
여하튼 이 은행이 Swiss Basel에 있다. 이거 이름을 따서 Basel II란 것을 만들었다. 이 룰의 이것의 핵심은 뭐냐? 좋은 기업이면 8%, 나쁜 기업이면 그보다 높게 BIS 비율을 정하라고 규정을 준 것이다. 즉 Risk-sensitive 한 것이다.
그렇게 하라고 준 식이 바로 아래의 식이다.

$$\Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{p}^{x}}{\sqrt{(1-p)}}\right)$$

- Phi =표준정규분포의 Cof
- PD = Probability of Default

  이거는 과거의 히스토리를 따라서 구한다.
  즉 부도기업이 높은 중소기업의 경우 5%를 집어넣으면 거의 20% 나온다. 이거 대비해서 BIS를 쌓아야 한다.
  그런데 이 식이 누구의 식이냐? 바로 볼튼의 식이다. 70년대에 생각했던 부도에 대한 식이다. 30년도 더 된 사람들인데 우리 살고 있는 세상에 영향을 주고 있다는 점이 오늘 배울 교훈이다.
  여하튼 내년부터 우리나라에서도 Basel II를 시행할 예정이다.
  이 식이 어떻게 나온 것인가는 중간고사 끝나고 나서부터 할 예정이다.

Other more general option pricing problems often seem immune to reduction to a simple formula. Instead, numerical procedures must be employed to value these more complex options. Michael Brennan and Eduardo Schwartz (1977) have provided many interesting results along these lines. However, their techniques are rather complicated and are not directly related to the economic structure of the problem. Our formulation, by its very construction, leads to an alternative numerical procedure that is both simpler, and for many purposes, computationally more efficient.

Section 6 introduces these numerical procedures and extends the model to include puts and calls on stocks that pay dividends. Section 7 concludes the paper by showing how the model can be generalized in other important ways and discussing its essential role in valuation by arbitrage methods.

2. The Basic Idea

Suppose the current price of a stock is $S = \$50$, and at the end of a period of time, its price must be either $S^* = \$25$ or $S^* = \$100$. A call on the stock is available with a strike price of $K = \$50$

옵션을 판 것과 같다. 그리고 K만큼을 long을 한 것이다. 따라서 Liability나 Equity는 것은 전부 Call/Put 옵션의 포트폴리오로 설명할 수 있다. 이 이야기를 좀 더 진전시키면, A_T가 실제로는 S_T에 해당한다는 의미도 된다. (즉 만기 시점의 가격)
즉 S_T를 설명한 모형으로 A_T를 설명하는 것도 가능하다는 것이다.

- **기업 가치**

  따라서 S_T는 아래의 stochastic 미방으로 가정할 수 있다.

  $$dS=\mu Sdt+\sigma SdW_t$$

  다만 sigma만 약간 의미가 다르다. 여기에서는 주식 시장의 변동성이 아닌 기업 가치의 변동성이 된다. 사실 Bolton의 이야기는 여기까지만이다. 그 이후에 사람들이 기업 가치의 변동성 구할 때, 기업의 가치를 가장 잘 반영하는 수치인 주식을 가지고 구하였다. 이를 $P(A_T<K)$를 가지고 구하였는데, 그것이 그 기업의 부도 확률이다. 따라서 그 기업의 부도 확률이 낮으면 낮을수록 높은 신용등급을 얻고, 높을수록 낮은 신용등급을 얻을 것이다.

  이런 신용등급을 해 주는 회사들이 S&P라든지 무디스라든지 하는 회사들이다. 이들은 Historical 한 자료들이 많다. (즉 AAA인 기업들이 어떻게 부도가 나느냐? 하는 등의 자료들을 이용한다)

- Binomial 모형을 어떻게 만들 것인가를 설명하는 것이다.

2. The Basic Idea

Suppose the current price of a stock is $S = \$50$, and at the end of a period of time, its price must be either $S^* = \$25$ or $S^* = \$100$. A call on the stock is available with a strike price of $K = \$50$, expiring at the end of the period.[3] It is also possible to borrow and lend at a 25% rate of interest. The one piece of information left unfurnished is the current value of the call, C. However, if riskless profitable arbitrage is not possible, we can deduce from the given information *alone* what the value of the call *must* be!

Consider the following levered hedge:

(1) write 3 calls at C each,
(2) buy 2 shares at \$50 each, and
(3) borrow \$40 at 25%, to be paid back at the end of the period.

Table 1 gives the return from this hedge for each possible level of the stock price at expiration. Regardless of the outcome, the hedge exactly breaks even on the expiration date. Therefore, to prevent profitable riskless arbitrage, its current cost must be zero; that is,

$$3C - 100 + 40 = 0$$

The current value of the call must then be $C = \$20$.

[3] To keep matters simple, assume for now that the stock will pay no cash dividends during the life of the call. We also ignore transaction costs, margin requirements and taxes.

- Binomial 모형을 어떻게 만들 것인가를 설명하는 것이다.
- Call option을 팔 때, C라는 가격을 가지고 파는 것을 말한다.
- r(무위험이자율)=25%
- 여하튼 이런 call option이 있다고 하자. 그럼 이 call option의 가격이 얼마인지 결정할 수 있다는 것이다. Arbitrage가 존재하지 않는 경우 이 주어진 것만 가지고도 call option의 가격을 계산할 수 있다.
- **[다음의 case]**
  1) Call option을 3개 팜
  2) $50까리 주식을 2개 삼.
  3) $40을 25%로 대출받음. (혹은 채권)

: Arbitrage가 존재하지 않기 위해서는 이들을 합친 값이 0이 되어야 한다. 이를 만족하는 C의 가격은 $20이다.

Table 1
Arbitrage Table Illustrating the Formation of a Riskless Hedge

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Arbitrage Table Illustrating the Formation of a Riskless Hedge

	present date	expiration date $S^* = \$25$	expiration date $S^* = \$100$
write 3 calls	$3C$	—	–150
buy 2 shares	–100	50	200
borrow	40	–50	–50
total		—	—

- 이는 강의자료 Excel 파일에 있던 거랑 비슷한 파일이다.

If the call were not priced at \$20, a sure profit would be possible. In particular, if $C = \$25$, the above hedge would yield a current cash inflow of \$15 and would experience no further gain or loss in the future. On the other hand, if $C = \$15$, then the same thing could be accomplished by buying 3 calls, selling short 2 shares, and lending \$40.

- 이들은 어떤 관계를 만족시켜야 하나?
- 미래에 항상 0이라는 profit이라고 하기 때문에 현재 시점의 가치도 0이 된다. (arbitrage가 존재하지 않는다고 가정했을 때) 즉 이걸 만족해야 하기 때문에 콜옵션가는 \$20이 되어야 한다.
- 그런데 만약 call option이 이것과 같지 않으면 어떻게 될까? Call option의 가격이 \$25이다? 그런 경우 하나를 팔 때마다 \$5의 이익을 낼 수 있다. 그리고 만기시점에는 이익도 손실도 없게 된다.

Table 1 can be interpreted as demonstrating that *an appropriately levered position in stock will replicate the future returns of a call.* That is, if we buy shares and borrow against them in the right proportion, we can, in effect, duplicate a pure position in calls. In view of this, it should seem less surprising that all we needed to determine the *exact* value of the call was its *strike price, underlying stock price, range of movement in the underlying stock price, and the rate of interest*. What may seem more incredible is what we do not need to know: among other things, *we do not need to know the probability that the stock price will rise or fall.* Bulls and bears must agree on the value of the call, relative to its underlying stock price!

This example is very simple, but it shows several essential features of option pricing. And we will soon see that it is not as unrealistic as it seems.

3. The Binomial Option Pricing Formula

In this section, we will develop the framework illustrated in the example into a complete valuation method. We begin by assuming that the stock price follows a multiplicative binomial process over discrete periods. The rate of return on the stock over each period can have two possible values: $u - 1$ with probability q, or $d - 1$ with probability $1 - q$. Thus, if the current stock price is S, the stock price at the end of the period will be either uS or dS. We can represent this movement with the following diagram:

uS with probability q

S

dS with probability $1 - q$

- 이는 유명한 그림이다. Q를 가지고 위로 올라가면 uS가 되고 내려가면 dS가 된다.

We also assume that the interest rate is constant. Individuals may borrow or lend as much as they wish at this rate. To focus on the basic issues, we will continue to assume that there are no

4

taxes, transaction costs, or margin requirements. Hence, individuals are allowed to sell short any security and receive full use of the proceeds.[4]

- 여기서 r이라고 하는 것은 gross rate, 즉 **[1+이자율]**이 된다.

Letting r denote one plus the riskless interest rate over one period, we require $u > r > d$. If these inequalities did not hold, there would be profitable riskless arbitrage opportunities involving only the stock and riskless borrowing and lending.[5]

: u가 제일 커야 되고, 그 다음에 r > d 이다. (이 r은 gross rate이다)
그런데 만약 r(gross rate)이 u보다 크면? 그럼 아무도 주식을 사지 않는다. 그 경우 주식 가진 사람은 주식을 short하고 은행에 돈을 넣어버리게 된다. 그 반대의 경우에는 은행에서 돈을 빌려서 모조리 다 주식에 투자해버리게 된다. 따라서 항상 이 관계가 성립해야만 한다.

To see how to value a call on this stock, we start with the simplest situation: the expiration date is just one period away. Let C be the current value of the call, C_u be its value at the end of the period if the stock price goes to uS and C_d be its value at the end of the period if the stock price goes to dS. Since there is now only one period remaining in the life of the call, we know that the terms of its contract and a rational exercise policy imply that $C_u = \max[0, uS - K]$ and $C_d = \max[0, dS - K]$. Therefore,

$$C \begin{cases} C_u = \max[0, uS - K] & \text{with probability } q \\ C_d = \max[0, dS - K] & \text{with probability } 1-q \end{cases}$$

: 이는 call option이다. 위쪽은 C_u, 아래쪽은 C_d라고 표시했다.

Suppose we form a portfolio containing Δ shares of stock and the dollar amount B in riskless bonds.[6] This will cost $\Delta S + B$. At the end of the period, the value of this portfolio will be

: Stock을 Δ만큼 사고, B만큼의 무위험 채권 포트폴리오를 구성했다고 할 경우 그 포트폴리오의 가치는 아래와 같다.

$$\Delta S + B \begin{cases} \Delta uS + rB & \text{with probability } q \\ \Delta dS + rB & \text{with probability } 1-q \end{cases}$$

이 경우 무위험채권이기 때문에 일정한 금액을 주게 된다.(이 금액은 rB이다)
그리고 이 때 r은 gross rate이다.
주식의 경우를 보면 upstate가 되어도 그 개수만큼을 반영해야 한다.즉 S가 uS로 변하든지 dS로 변하든지 이를 반영해야 한다.

Since we can select Δ and B in any way we wish, suppose we choose them to equate the end-of-period values of the portfolio and the call for each possible outcome. This requires that

$$\Delta uS + rB = C_u$$
$$\Delta dS + rB = C_d$$

따라서 Call option의 경우 옆과 같이 가격을 정할 수 있다.

Solving these equations, we find

$$\Delta = \frac{C_u - C_d}{(u-d)S}, \quad B = \frac{uC_d - dC_u}{(u-d)r} \qquad (1)$$

이 식은 결국 Delta hedge의 식과 궁극적으로 같다.

교수님이 설명했던 방식과는 약간 다르다. 교수님은 upstate가 되건 downstate이건 같게 만드는 Δ를 찾는다고 했는데, 이 사람들은 down일 때는 down의 가치를 갖는 포트폴리오를 찾고 up일 때는 up의 가치를 갖는 포

[4] Of course, restitution is required for payouts made to securities held short.

[4] Of course, restitution is required for payouts made to securities held short.

[5] We will ignore the uninteresting special case where q is zero or one and $u = d = r$.

[6] Buying bonds is the same as lending; selling them is the same as borrowing.

교수님이 설명했던 방식과는 약간 다르다. 교수님은 upstate가 되건 downstate이건 같게 만드는 Δ를 찾는다고 했는데, 이 사람들은 down일 때는 down의 가치를 갖는 포트폴리오를 찾고 up일 때는 up의 가치를 갖는 포트폴리오를 찾는다. 여하튼 이렇게 구해서 연립방정식을 구한다. 그리고 이를 대입한 것이 call option의 현재 시점에서의 가격이 되어야만 한다는 것이다.

With Δ and B chosen in this way, we will call this the hedging portfolio.

If there are to be no riskless arbitrage opportunities, the current value of the call, C, cannot be less than the current value of the hedging portfolio, $\Delta S + B$. If it were, we could make a riskless profit with no net investment by buying the call and selling the portfolio. It is tempting to say that it also cannot be worth more, since then we would have a riskless arbitrage opportunity by reversing our procedure and selling the call and buying the portfolio. But this overlooks the fact that the person who bought the call we sold has the right to exercise it immediately.

Suppose that $\Delta S + B < S - K$. If we try to make an arbitrage profit by selling calls for more than $\Delta S + B$, but less than $S - K$, then we will soon find that we are the source of arbitrage profits rather than the recipient. Anyone could make an arbitrage profit by buying our calls and exercising them immediately.

We might hope that we will be spared this embarrassment because everyone will somehow find it advantageous to hold the calls for one more period as an investment rather than take a quick profit by exercising them immediately. But each person will reason in the following way. If I do not exercise now, I will receive the same payoff as a portfolio with ΔS in stock and B in bonds. If I do exercise now, I can take the proceeds, $S - K$, buy this same portfolio and some extra bonds as well, and have a higher payoff in every possible circumstance. Consequently, no one would be willing to hold the calls for one more period.

Summing up all of this, we conclude that if there are to be no riskless arbitrage opportunities, it must be true that

$$C = \Delta S + B = \frac{C_u - C_d}{u-d} + \frac{uC_d - dC_u}{(u-d)r} = \left[\left(\frac{r-d}{u-d}\right)C_u + \left(\frac{u-r}{u-d}\right)C_d\right] / r \qquad (2)$$

: 합쳐서 계산하면 이런 형태가 나온다. 이 중 앞에 있는 Term을 P 즉 upstate로 놓고 뒤를 downstate로 놓으면 된다.

if this value is greater than $S - K$ and if not, $C = S - K$.[7]

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if this value is greater than $S - K$, and if not, $C = S - K$.[7]

Equation (2) can be simplified by defining

$$\boxed{p \equiv \frac{r-d}{u-d}} \quad \text{and} \quad \boxed{1 - p \equiv \frac{u-r}{u-d}}$$

so that we can write

$$\underline{C = [pC_u + (1-p)C_d]/r} \qquad (3)$$

It is easy to see that in the present case, with no dividends, this will always be greater than $S - K$ as long as the interest rate is positive. To avoid spending time on the unimportant situations where the interest rate is less than or equal to zero, we will now assume that r is always greater

[7] In some applications of the theory to other areas, it is useful to consider options that can be exercised only on the expiration date. These are usually termed European options. Those that can be exercised at any earlier time as well, such as we have been examining here, are then referred to as American options. Our discussion could be easily modified to include European calls. Since immediate exercise is then precluded, their values would always be given by (2), even if this is less than $S - K$.

: 합쳐서 계산하면 이런 형태가 나온다. 이 중 앞에 있는 Term을 P 즉 upstate로 놓고 뒤를 downstate로 놓으면 된다.

그런데 이 때 r은 무위험 이자율이다. 이를 discount 해 주면 콜옵션의 가치가 되는 것이다.

즉 저자는 모든 것을 다 discrete 하게 생각한 것이다. 그리고 결론은 우리가 했던 것과 똑같은 결론이 나온다!!

than one. Hence, (3) is the exact formula for the value of a call one period prior to the expiration in terms of S, K, u, d, and r.

To confirm this, note that if $uS \le K$, then $S < K$ and $C = 0$, so $C > S - K$. Also, if $dS \ge K$, then $C = S - (K/r) > S - K$. The remaining possibility is $uS > K > dS$. In this case, $C = p(uS - K)/r$. This is greater than $S - K$ if $(1-p)dS > (p - r)K$, which is certainly true as long as $r > 1$.

<u>This formula has a number of notable features.</u> First, the probability q does not appear in the formula. This means, surprisingly, that even if different investors have different subjective probabilities about an upward or downward movement in the stock, they could still agree on the relationship of C to S, u, d, and r.

Second, the value of the call does not depend on investors' attitudes toward risk. In constructing the formula, the only assumption we made about an individual's behavior was that he prefers more wealth to less wealth and therefore has an incentive to take advantage of profitable riskless arbitrage opportunities. We would obtain the same formula whether investors are risk-averse or

- **유의해야 할 성질들**
  1) q라고 하는 숫자 자체는 formula에 나타나지 않는다.
     - 즉 이는 investor가 어떤 생각을 가지고 있는지 관계없이 우리가 C라고 하는 값을 결정할 수 있다는 것이다.
  2) Risk를 선호하는 사람이든지 회피하는 사람이든지 전혀 상관없이 똑같이 같은 공식을 적용할 수 있다.
  3) 콜옵션 계산할 때 유의하게 영향을 주는 것은 자기 기초자산 하나

Second, the value of the call does not depend on investors' attitudes toward risk. In constructing the formula, the only assumption we made about an individual's behavior was that he prefers more wealth to less wealth and therefore has an incentive to take advantage of profitable riskless arbitrage opportunities. We would obtain the same formula whether investors are risk-averse or risk-preferring.

Third, the only random variable on which the call value depends is the stock price itself. In particular, it does not depend on the random prices of other securities or portfolios, such as the market portfolio containing all securities in the economy. If another pricing formula involving other variables was submitted as giving equilibrium market prices, we could immediately show that it was incorrect by using our formula to make riskless arbitrage profits while trading at those prices.

It is easier to understand these features if it is remembered that the formula is only a relative pricing relationship giving C in terms of S, u, d, and r. Investors' attitudes toward risk and the characteristics of other assets may indeed influence call values indirectly, through their effect on these variables, but they will not be separate determinants of call value.

Finally, observe that $p \equiv (r-d)/(u-d)$ is always greater than zero and less than one, so it has the properties of a probability. In fact, p is the value q would have in equilibrium if investors were risk-neutral. To see this, note that the expected rate of return on the stock would then be the riskless interest rate, so

$$q(uS) + (1-q)(dS) = rS$$

and

$$q = (r-d)/(u-d) = p$$

Hence, the value of the call can be interpreted as the expectation of its discounted future value in a risk-neutral world. In light of our earlier observations, this is not surprising. Since the formula does not involve q or any measure of attitudes toward risk, then it must be the same for any set of preferences, including risk neutrality.

It is important to note that this does not imply that the equilibrium expected rate of return on the call is the riskless interest rate. Indeed, our argument has shown that, in equilibrium, holding the call over the period is exactly equivalent to holding the hedging portfolio. Consequently, the risk

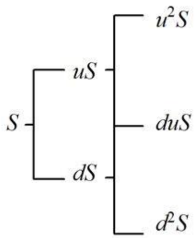
and expected rate of return of the call must be the same as that of the hedging portfolio. It can be shown that $\Delta \geq 0$ and $B \leq 0$, so the hedging portfolio is equivalent to a particular levered long

2) Risk를 선호하는 사람이든지 회피하는 사람이든지 전혀 상관없이 똑같이 같은 공식을 적용할 수 있다.
3) 콜옵션 계산할 때 유일하게 영향을 주는 것은 자기 기초자산 하나밖에 없다.

- **[참고]**
  마켓 포트폴리오란 것이 있다. 다른 놈 가격이 어떻게 되든 상관없이 가지고 있어야 한다는 의미이다.
- 또한 옵션은 다른 애들에 의해서 상대적으로 구해진다. (단 이를 모르는 상태에서 가격을 결정할 수는 없다) 즉 다른 애들의 가격을 통해서 자신의 가격을 결정해 나가면 된다.
- q는 실제 어떤 일이 일어날 확률이다. 그런데 p라는 확률은 risk neutral 한 사람이 세상을 바라볼 확률이 바로 q가 되는 것이다. 이렇게 계산한 formula는 investor가 risk를 좋아하건 싫어하건 관계없이 받아들일 수 밖에 없는 가격이다. 그래서 p를 risk-neutral이라고 부르는 것이다.
- **[메모] 좀 더 자세히 읽어보자!!!**

and expected rate of return of the call must be the same as that of the hedging portfolio. It can be shown that $\Delta \geq 0$ and $B \leq 0$, so the hedging portfolio is equivalent to a particular levered long position in the stock. In equilibrium, the same is true for the call. Of course, if the call is currently mispriced, its risk and expected return over the period will differ from that of the hedging portfolio.

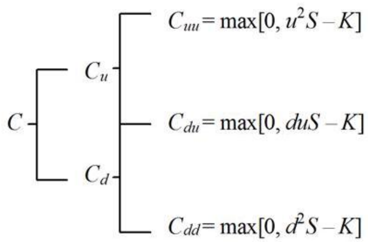
Now we can consider the next simplest situation: a call with two periods remaining before its expiration date. In keeping with the binomial process, the stock can take on three possible values after two periods,



- 이제 2-step으로 나간 경우이다. 그림만 보아도 어떻게 하는지 다 알 것 이다.

  여하튼 기초자산의 가격이 움직이는 것을 나타내고 있다.

Similarly, for the call,



- 이 그림에서 위의 부분만 우선 보면, one-step binomial tree처럼 생겼다.

C_{uu} stands for the value of a call two periods from the current time if the stock price moves upward each period; C_{du} and C_{dd} have analogous definitions.

At the end of the current period there will be one period left in the life of the call, and we will be faced with a problem identical to the one we just solved. Thus, from our previous analysis, we know that when there are two periods left,

$$C_u = [pC_{uu} + (1-p)C_{ud}]/r$$

and

$$C_d = [pC_{du} + (1-p)C_{dd}]/r \tag{4}$$

Again, we can select a portfolio with ΔS in stock and B in bonds whose end-of-period value will be C_u if the stock price goes to uS and C_d if the stock price goes to dS. Indeed, the functional form of Δ and B remains unchanged. To get the new values of Δ and B, we simply use equation (1) with the new values of C_u and C_d.

Can we now say, as before, that an opportunity for profitable riskless arbitrage will be available if the current price of the call is not equal to the new value of this portfolio or $S - K$, whichever is greater? Yes, but there is an important difference. With one period to go, we could plan to lock in a riskless profit by selling an overpriced call and using part of the proceeds to buy the hedging portfolio. At the end of the period, we knew that the market price of the call must be equal to the value of the portfolio, so the entire position could be safely liquidated at that point. But this was true only because the end of the period was the expiration date. Now we have no such guarantee. At the end of the current period, when there is still one period left, the market price of the call could still be in disequilibrium and be greater than the value of the hedging portfolio. If we closed out the position then, selling the portfolio and repurchasing the call, we could suffer a loss that would more than offset our original profit. However, we could always avoid this loss by maintaining the portfolio for one more period. The value of the portfolio at the end of the current period will always be exactly sufficient to purchase the portfolio we would want to hold over the last period. In effect, we would have to readjust the proportions in the hedging portfolio, but we would not have to put up any more money.

Consequently, we conclude that even with two periods to go, there is a strategy we could follow which would guarantee riskless profits with no net investment if the current market price of a call differs from the maximum of $\Delta S + B$ and $S - K$. Hence, the larger of these is the current value of the call.

Since Δ and B have the same functional form in each period, the current value of the call in terms of C_u and C_d will again be $C = [pC_u + (1-p)C_d]/r$ if this is greater than $S - K$, and $C = S - K$ otherwise. By substituting from equation (4) into the former expression, and noting that $C_{du} = C_{ud}$, we obtain

$$\boxed{\begin{aligned} C &= [p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}]/r^2 \\ &= [p^2 \max[0, u^2 S - K] + 2p(1-p)\max[0, duS - K] + (1-p)^2 \max[0, d^2 S - K]]/r^2 \end{aligned}} \quad (5)$$

- Two -step binomial tree에서의 가격.

A little algebra shows that this is always greater than $S - K$ if, as assumed, r is always greater than one, so this expression gives the exact value of the call.[8]

All of the observations made about formula (3) also apply to formula (5), except that the number of periods remaining until expiration, n, now emerges clearly as an additional determinant of the call value. For formula (5), $n = 2$. That is, the full list of variables determining C is S, K, n, u, d, and r.

[8] In the current situation, with no dividends, we can show by a simple direct argument that if there are no arbitrage

d, and r.

[8] In the current situation, with no dividends, we can show by a simple direct argument that if there are no arbitrage opportunities, then the call value must always be greater than $S - K$ before the expiration date. Suppose that the call is selling for $S - K$. Then there would be an easy arbitrage strategy that would require no initial investment and would always have a positive return. All we would have to do is buy the call, short the stock, and invest K dollars in bonds. See Merton (1973). In the general case, with dividends, such an argument is no longer valid, and we must use the procedure of checking every period.

We now have a recursive procedure for finding the value of a call with any number of periods to go. By starting at the expiration date and working backwards, we can write down the general valuation formula for any n:

$$C = \left[\sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} \max[0, u^j d^{n-j} S - K] \right] / r^n \qquad (6)$$

- 위의 콜옵션을 n-step에 대해서 generalize 시키면 이렇게된다.

This gives us the complete formula, but with a little additional effort we can express it in a more convenient way.

Let a stand for the minimum number of upward moves that the stock must make over the next n periods for the call to finish in-the-money. Thus a will be the smallest non-negative integer such that $u^a d^{n-a} S > K$. By taking the natural logarithm of both sides of this inequality, we could write a as the smallest non-negative integer greater than $\boxed{\log(K/Sd^n)/\log(u/d)}$.

- a라는 숫자를 놓고 보자. 동전의 앞면이 몇 번 나와야만 max 안에 있는 것이 0보다는 큰 숫자가 나올 것인가? 여하튼 이를 위한 최소한의 동전의 개수가 바로 a이다.
- 이 식을 대해서 놓고 a에 대해서 구하면, **log(K/Sd^n)/log(u/d)**가 된다. 따라서 이거보다 큰 정수가 되면 된다는 의미이다. 이렇게 a를 정의해 놓고 나면 0이 max로 나올 것이기 때문에 아래와 같은 식이 된다.
- 즉 이 때 j=(a~n) 까지만 summation 하면 된다.

For all $j < a$,

$$\max[0, u^j d^{n-j} S - K] = 0$$

and for all $j \geq a$,

$$\max[0, u^j d^{n-j} S - K] = u^j d^{n-j} S - K$$

Therefore,

$$\boxed{C = \left[\sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} [u^j d^{n-j} S - K] \right] / r^n}$$

Of course, if $a > n$, the call will finish out-of-the-money even if the stock moves upward every period, so its current value must be zero.

Of course, if $a > n$, the call will finish out-of-the-money even if the stock moves upward every period, so its current value must be zero.

By breaking up C into two terms, we can write

$$C = S\left[\sum_{j=a}^{n}\left(\frac{n!}{j!(n-j)!}\right)p^j(1-p)^{n-j}\left(\frac{u^j d^{n-j}}{r^n}\right)\right] - Kr^{-n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!(n-j)!}\right)p^j(1-p)^{n-j}\right]$$

- 이는 이항 누적 분포와 똑같다.
- 반면 앞쪽은 야간 다르다. 뒤에 뭔가가 더 달라붙어있다. 이 부분을 집어넣어보면,

$$\frac{n!}{j!(n-j)!}p^j(1-p)^{n-j}\left(\frac{u^j d^{n-j}}{r^n}\right) = \frac{n!}{j!(n-j)!}\frac{(up)^j}{r}\frac{((1-p)d)^{n-j}}{r}$$

가 된다.

이의 항들을 p'=(u/r)p로 정의하여 치환하여보자. 그럼 식의 우변의 경우 전체적으로 u/r *p', d/r *(1-p')의 형태가 된다. 이걸 더해서 1이 된다는 것을 확인해 보면 되는 것이다.

Now, the latter bracketed expression is the complementary binomial distribution function $\phi[a; n, p]$. The first bracketed expression can also be interpreted as a complementary binomial distribution function $\phi[a; n, p']$, where

p'

$$p' \equiv (u/r)p \quad \text{and} \quad 1-p' \equiv (d/r)(1-p)$$

p' is a probability, since $0 < p' < 1$. To see this, note that $p < (r/u)$ and

$$p^j(1-p)^{n-j}\left(\frac{u^j d^{n-j}}{r^n}\right) = \left[\frac{u}{r}p\right]^j = \left[\frac{d}{r}(1-p)\right]^{n-j} = p'^j(1-p')^{n-j}$$

- 이걸 실제로 더해서 계산해 보면 1이 된다.

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In summary:

Binomial Option Pricing Formula

$$C = S\phi[a; n, p'] - Kr^{-n}\phi[a; n, p]$$

where

$p \equiv (r-d)/(u-d)$ and P' $\cancel{p'} \equiv (u/r)p$

$a \equiv$ the smallest non-negative integer

- **[매우 중요] 이는 새로운 notation이다. 무슨 내용인가? 이는 시험문제에 나온다. 이것이 뭔지 정확하게 이해하고 있어야 문제를 풀 수 있다.**

$$\Phi[a; n, p]$$

앞면이 나올 확률이 p인 확률을 n번 던졌을 때 a부터 시작해서 오른쪽 확률을 모두 더하는 경우이다. 즉 이를 풀어서 쓰면,

$$p \equiv (r-d)/(u-d) \text{ and } \cancel{p}\,\text{P'} \equiv (u/r)p$$

$a \equiv$ the smallest non-negative integer greater than $\log(K/Sd^n)/\log(u/d)$

If $a > n$, then $C = 0$.

앞면이 나올 확률이 p인 확률을 n번 던졌을 때 a부터 시작해서 오른쪽 확률을 모두 더하는 경우이다. 즉 이를 풀어서 쓰면,

$$\Phi[a;n,p] = \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

이다.

a는 최소한 몇 번의 앞면이 나와야만 되는가? 지금 주가가 100인데 행사 가격이 500이라고 해 보자. 이 경우 주가가 아무리 올라봤자 쓸모가 없다. **즉 a가 총 동전을 던질 수 있는 횟수보다도 크게 나오면 아무리 동전을 던져봤자 그 숫자에 도달할 수 없으므로 그 경우 콜옵션의 가격은 0이 된다는 의미이다.**

It is now clear that all of the comments we made about the one period valuation formula are valid for any number of periods. In particular, the value of a call should be the expectation, in a risk-neutral world, of the discounted value of the payoff it will receive. In fact, that is exactly what equation (6) says. Why, then, should we waste time with the recursive procedure when we can write down the answer in one direct step? The reason is that while this one-step approach is always technically correct, it is really useful only if we know in advance the circumstances in which a rational individual would prefer to exercise the call before the expiration date. If we do not know this, we have no way to compute the required expectation. In the present example, a call on a stock paying no dividends, it happens that we can determine this information from other sources: the call should never be exercised before the expiration date. As we will see in section 6, with puts or with calls on stocks that pay dividends, we will not be so lucky. Finding the optimal exercise strategy will be an integral part of the valuation problem. The full recursive procedure will then be necessary.

For some readers, an alternative "complete markets" interpretation of our binomial approach may be instructive. Suppose that π_u and π_d represent the state-contingent discount rates to states u and d, respectively. Therefore, π_u would be the current price of one dollar received at the end of the period, if and only if state u occurs. Each security —a riskless bond, the stock, and the option —must all have returns discounted to the present by π_u and π_d if no riskless arbitrage opportunities are available. Therefore,

: state price를 가지고 설명할 수 있다는 의미이다. πᵤ와 π_d가 바로 state price이다.
즉 아래와 같이 가정하는 것이다.

- $\pi_u = 1 \mid 0$
- $\pi_d = 0 \mid 1$

$$\boxed{\begin{aligned} 1 &= \pi_u r + \pi_d r \\ S &= \pi_u(uS) + \pi_d(dS) \end{aligned}}$$
$$C = \pi_u C_u + \pi_d C_d$$

- R은 gross rate이다. 그런데 이의 가격을 계산해 보자?
  1 = r | r 인 상품의 가격을 π_u π_d를 이용해서 계산을 한 것이다.
- 역시 비슷하게 S = uS | dS 를 π_u π_d 를 이용해서 계산을 한 것이다.
- 세번째 것도 비슷하다.
- 나머지 모든 숫자를 안다고 가정하면 이 연립방정식을 풀 수 있다.

The first two equations, for the bond and the stock, imply

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$$\pi_u = \left(\frac{r-d}{u-d}\right)\frac{1}{r} \quad \text{and} \quad \pi_d = \left(\frac{u-r}{u-d}\right)\frac{1}{r}$$

Substituting these equalities for the state-contingent prices in the last equation for the option yields equation (3).

It is important to realize that we are not assuming that the riskless bond and the stock and the option are the only three securities in the economy, or that other securities must follow a binomial process. Rather, however these securities are priced in relation to others in equilibrium, among themselves they must conform to the above relationships.

From either the hedging or complete markets approaches, it should be clear that three-state or trinomial stock price movements will not lead to an option pricing formula based solely on arbitrage considerations. Suppose, for example, that over each period the stock price could move to uS or dS or remain the same at S. A choice of Δ and B that would equate the returns in two states could not in the third. That is, a riskless arbitrage position could not be taken. Under the complete markets interpretation, with three equations in now three unknown state-contingent prices, we would lack the redundant equation necessary to price one security in terms of the other two.

4. Riskless Trading Strategies

The following numerical example illustrates how we could use the formula if the current *market price* M ever diverged from its *formula value* C. If $M > C$, we would hedge, and if $M < C$, "reverse hedge", to try and lock in a profit. Suppose the values of the underlying variables are

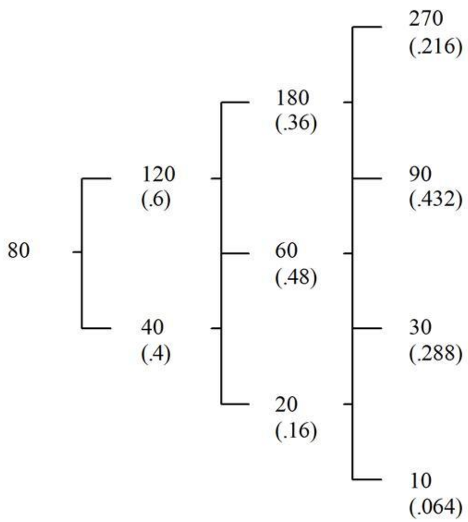
$$S = 80, \quad n = 3, \quad K = 80, \quad u = 1.5, \quad d = 0.5, \quad r = 1.1$$

In this case, $p = (r - d)/(u - d) = 0.6$. The relevant values of the discount factor are

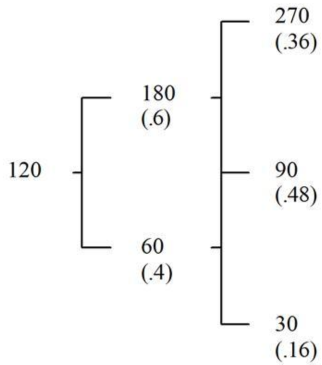
$$r^{-1} = 0.909, \quad r^{-2} = 0.826, \quad r^{-3} = 0.751$$

The paths the stock price may follow and their corresponding probabilities (using probability p) are, when $n = 3$, with $S - 80$,

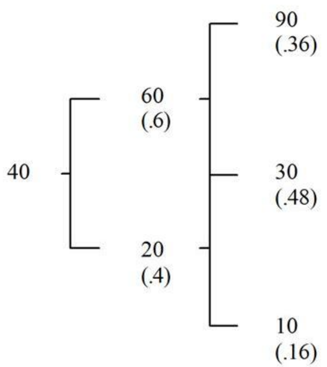
- 그것을 풀어놓은 것이 바로 이거다. 이렇게 구한 π_u와 π_d를 가지고 3번째 식, option price formula에 집어넣으면 바로 옵션 프라이싱 식이 되는 것이다. (6 page에 있음)



when $n = 2$, if $S = 120$,



when $n = 2$, if $S = 40$,



Using the formula, the current value of the call would be

$$C = 0.751[0.064(0) + 0.288(0) + 0.432(90 - 80) + 0.216(270 - 80)] = 34.065.$$

Recall that to form a riskless hedge, for each call we sell, we buy and subsequently keep adjusted a portfolio with ΔS in stock and B in bonds, where $\Delta = (C_u - C_d)/(u - d)S.$ The following tree diagram gives the paths the call value may follow and the corresponding values of Δ:

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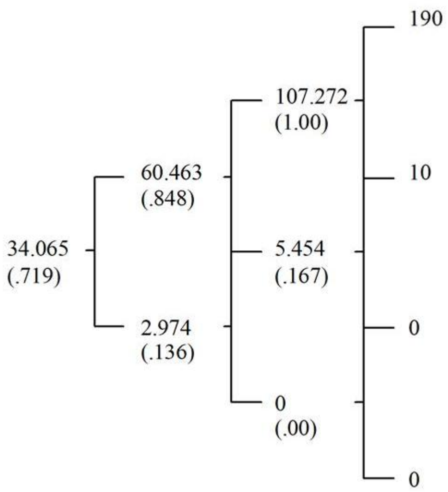

With this preliminary analysis, we are prepared to use the formula to take advantage of mispricing in the market. Suppose that when $n = 3$, the market price of the call is 36. Our formula tells us the call should be worth 34.065. The option is overpriced, so we could plan to sell it and assure ourselves of a profit equal to the mispricing differential. Here are the steps you could take for a typical path the stock might follow.

Step 1 ($n = 3$): Sell the call for 36. Take 34.065 of this and invest it in a portfolio containing $\Delta = 0.719$ shares of stock by borrowing 0.719(80) – 34.065 = 23.455. Take the remainder, 36 – 34.065 = 1.935, and put it in the bank.

Step 2 ($n = 2$): Suppose the stock goes to 120 so that the new Δ is 0.848. Buy 0.848 – 0.719 = 0.129 more shares of stock at 120 per share for a total expenditure of 15.480. Borrow to pay the

34.065 = 1.935, and put it in the bank.

Step 2 ($n = 2$): Suppose the stock goes to 120 so that the new Δ is 0.848. Buy 0.848 – 0.719 = 0.129 more shares of stock at 120 per share for a total expenditure of 15.480. Borrow to pay the bill. With an interest rate of 0.1, you already owe 23.455(1.1) = 25.801. Thus, your total current indebtedness is 25.801 + 15.480 = 41.281.

Step 3 ($n = 1$): Suppose the stock price now goes to 60. The new Δ is 0.167. Sell 0.848 – 0.167 = 0.681 shares at 60 per share, taking in 0.681(60) = 40.860. Use this to pay back part of your borrowing. Since you now owe 41.281(1.1) = 45.409, the repayment will reduce this to 45.409 – 40.860 = 4.549.

Step 4d ($n = 0$): Suppose the stock price now goes to 30. The call you sold has expired worthless. You own 0.167 shares of stock selling at 30 per share, for a total value of 0.167(30) = 5. Sell the stock and repay the 4.549(1.1) = 5 that you now owe on the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.341$.

Step 4u ($n = 0$): Suppose, instead, the stock price goes to 90. The call you sold is in the money at the expiration date. Buy back the call, or buy one share of stock and let it be exercised, incurring a loss of 90 – 80 = 10 either way. Borrow to cover this, bringing your current indebtedness to 5 + 10 = 15. You own 0.167 shares of stock selling at 90 per share, for a total value of 0.167(90) = 15. Sell the stock and repay the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.341$.

In summary, if we were correct in our original analysis about stock price movements (which did not involve the unenviable task of predicting whether the stock price would go up or down), and if we faithfully adjust our portfolio as prescribed by the formula, then we can be assured of walking away in the clear at the expiration date, while still keeping the original differential and the interest it has accumulated. It is true that closing out the position before the expiration date, which involves buying back the option at its then current market price, might produce a loss which would more than offset our profit, but this loss could always be avoided by waiting until the expiration date. Moreover, if the market price comes into line with the formula value before the expiration date, we can close out the position then with no loss and be rid of the concern of keeping the portfolio adjusted.

It still might seem that we are depending on rational behavior by the person who bought the call we sold. If instead he behaves foolishly and exercises at the wrong time, could he makes things worse for us as well as for himself? Fortunately, the answer is no. Mistakes on his part can only mean greater profits for us. Suppose that he exercises too soon. In that circumstance, the hedging portfolio will always be worth more than $S-K$, so we could close out the position then with an extra profit.

Suppose, instead, that he fails to exercise when it would be optimal to do so. Again there is no problem. Since exercise is now optimal, our hedging portfolio will be worth $S-K$.[9] If he had exercised, this would be exactly sufficient to meet the obligation and close out the position. Since he did not, the call will be held at least one more period, so we calculate the new values of C_u and C_d and revise our hedging portfolio accordingly. But now the amount required for the portfolio, $\Delta S + B$, is less than the amount we have available, $S-K$. We can withdraw these extra profits now and still maintain the hedging portfolio. The longer the holder of the call goes on making mistakes, the better off we will be.

Consequently, we can be confident that things will eventually work out right no matter what the other party does. The return on our total position, when evaluated at prevailing market prices at intermediate times, may be negative. But over a period ending no later than the expiration date, it will be positive.

In conducting the hedging operation, the essential thing was to maintain the proper proportional relationship: for each call we are short, we hold Δ shares of stock and the dollar amount B in bonds in the hedging portfolio. To emphasize this, we will refer to the number of shares held for each call as the hedge ratio. In our example, we kept the number of calls constant and made adjustments by buying or selling stock and bonds. As a result, our profit was independent of the market price of the call between the time we initiated the hedge and the expiration date. If things got worse before they got better, it did not matter to us.

Instead, we could have made the adjustments by keeping the number of shares of stock constant and buying or selling calls and bonds. However, this could be dangerous. Suppose that after initiating the position, we needed to increase the hedge ratio to maintain the proper proportions. This can be achieved in two ways:

(a) buy more stock, or
(b) buy back some of the calls.

If we adjust through the stock, there is no problem. If we insist on adjusting through the calls, not only is the hedge no longer riskless, but it could even end up losing money! This can happen if the call has become even more overpriced. We would then be closing out part of our position in calls at a loss. To remain hedged, the number of calls we would need to buy back depends on their value, not their price. Therefore, since we are uncertain about their price, we then become uncertain about the return from the hedge. Worse yet, if the call price gets high enough, the loss on the closed portion of our position could throw the hedge operation into an overall loss.

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[9] If we were reverse hedging by buying an undervalued call and selling the hedging portfolio, then we would ourselves want to exercise at this point. Since we will receive $S-K$ from exercising, this will be exactly enough money to buy back the hedging portfolio.

To see how this could happen, let us rerun the hedging operation, where we adjust the hedge ratio by buying and selling calls.

Step 1 ($n = 3$): Same as before.

Step 2 ($n = 2$): Suppose the stock goes to 120, so that the new $\Delta = 0.848$. The call price has gotten further out of line and is now selling for 75. Since its value is 60.463, it is now overpriced by 14.537. With 0.719 shares, you must buy back $1 - 0.848 = 0.152$ calls to produce a hedge ratio of $0.848 = 0.719/0.848$. This costs $75(0.152) = 11.40$. Borrow to pay the bill. With the interest rate of 0.1, you already owe $23.455(1.1) = 25.801$. Thus, your total current indebtedness is $25.801 + 11.40 = 37.201$.

Step 3 ($n = 1$): Suppose the stock goes to 60 and the call is selling for 5.454. Since the call is now fairly valued, no further excess profits can be made by continuing to hold the position. Therefore, liquidate by selling your 0.719 shares for $0.719(60) = 43.14$ and close out the call position by buying back 0.848 calls for $0.848(5.454) = 4.625$. This nets $43.14 - 4.625 = 38.515$. Use this to pay back part of your borrowing. Since you now owe $37.20(1.1) = 40.921$, after repayment you owe 2.406. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^2 = 2.341$. Unfortunately, after using this to repay your emaining borrowing, you still owe 0.065.

Since we adjusted our position at Step 2 by buying overpriced calls, our profit is reduced. Indeed, since the calls were considerably overpriced, we actually lost money despite apparent profitability of the position at Step 1. We can draw the following adjustment rule from our experiment: *To adjust a hedged position, never buy an overpriced option or sell an underpriced option.* As a corollary, whenever we can adjust a hedged position by buying more of an underpriced option or selling more of an overpriced option, our profit will be enhanced if we do so. For example, at Step 3 in the original hedging illustration, had the call still been overpriced, it would have been better to adjust the position by selling more calls rather than selling stock. In summary, by choosing the right side of the position to adjust at intermediate dates, *at a minimum* we can be assured of earning the original differential and its accumulated interest, and we may earn considerably more.

5. Limiting Cases

In reading the previous sections, there is a natural tendency to associate with each period some particular length of calendar time, perhaps a day. With this in mind, you may have had two

In reading the previous sections, there is a natural tendency to associate with each period some particular length of calendar time, perhaps a day. With this in mind, you may have had two objections. In the first place, prices a day from now may take on many more than just two possible values. Furthermore, the market is not open for trading only once a day, but, instead, trading takes place almost continuously.

These objections are certainly valid. Fortunately, our option pricing approach has the flexibility to meet them. Although it might have been natural to think of a period as one day, there was nothing that forced us to do so. We could have taken it to be a much shorter interval —say an hour —or even a minute. By doing so, we have met both objections simultaneously. Trading

[강의노트]

- 하루에 S = uS | dS 가 되어 버릴 수가 있나? 보통 주가 변동은 이것보다 훨씬 더 많은 값을 가지게 된다. 또한 하루에 한 번 밖에 trading을 못한다고 가정하였는데, 실제로 우리는 거의 continuous하게(매분 매초) trading 할 수 있다. 각각의 시간에서 360 step이면 하루가 흐른다. 지금 10만원 하는 주식이 오늘 하루가 지나면 361개의 다른 값을 가지도록 모델링이 가능한 것이다.
- 우리나라를 보면 모든 값을 다 가지고 있지는 않다. 일반적으로 소수점 이하 2번째 까지 쓰는데, 바이노미얼을 잘게 짜르면 소수점 이하 3번째까지도 자를 수 있다.

would take place far more frequently, and the stock price could take on hundreds of values by the end of the day.

However, if we do this, we have to make some other adjustments to keep the probability small that the stock price will change by a large amount over a minute. We do not want the stock to have the same percentage up and down moves for one minute as it did before for one day. But again there is no need for us to have to use the same values. We could, for example, think of the price as making only a very small percentage change over each minute.

To make this more precise, suppose that h represents the elapsed time between successive stock price changes. That is, if t is the fixed length of calendar time to expiration, and n is the number of periods of length h prior to expiration, then

$$h \equiv t/n$$

As trading takes place more and more frequently, h gets closer and closer to zero. We must then adjust the interval-dependent variables r, u, and d in such a way that we obtain empirically realistic results as h becomes smaller, or, equivalently, as $n \to \infty$.

When we were thinking of the periods as having a fixed length, r represented both the interest rate over a fixed length of calendar time and the interest rate over one period. Now we need to make a distinction between these two meanings. We will let r continue to mean one plus the interest rate over a fixed length of calendar time. When we have occasion to refer to one plus the interest rate over a period (trading interval) of length h, we will use the symbol $\hat{r}$.

- 주가 변동을 일할이 아닌 시할, 혹은 분할 계산할 경우 시간 간격에 따라 이를 조정해 줄 필요가 있다.
  따라서 **h = 한 스텝의 시간, t=만기까지의 시간, n=스텝의 수**
  로 정의할 수 있다.
- 즉 전체 t를 n개로 자르면 각각의 거리는 h가 된다. 근데 이 부분은 잘 읽어야 한다. 이거 헷갈리기 시작하면 정말 어렵다. (뒷부분에 비슷한 notation이 계속 나오기 때문)
  h는 한 스텝의 시간이기 때문에 trade가 점점 더 자주 일어나면 일어날수록 0에 가까워질 것이다. 그리고 지속적으로 u, d를 곱해나가면 맨 끝에 u^{100}, d^{100} 이런 식의 숫자가 나올 수 있다. 이를 방지하기 위해 u 와 r을 조정해 줄 필요가 있다.
- **r^ 의 정의**
  이는 매 스텝의 EAR이라고 생각하면 된다(r은 APR이라고 생각하라. Gross rate임에 유의할 것)
  이 경우 전체의 return이 얼마인가를 계산하기 위해서는 return들을 쭉 누적해서 계산

make a distinction between these two meanings. We will let r continue to mean one plus the interest rate over a fixed length of calendar time. When we have occasion to refer to one plus the interest rate over a period (trading interval) of length h, we will use the symbol $\hat{r}$.

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이 경우 전체의 return이 얼마인가를 계산하기 위해서는 return들을 쭉 누적해서 계산하면 되는데, 끝까지 다 오면 $(r\wedge)^n$ 이 된다.

Clearly, the size of $\hat{r}$ depends on the number of subintervals, n, into which t is divided. Over the n periods until expiration, the total return is $\hat{r}^n$, where $n = t/h$. Now not only do we want $\hat{r}$ to depend on n, but we want it to depend on n in a particular way —so that as n changes the total return $\hat{r}^n$ over the fixed time t remains the same. This is because the interest rate obtainable over some fixed length of calendar time should have nothing to do with how we choose to think of the length of the time interval h.

따라서 r^은 n에 의존한다. 매 period에 적용되는 return 이자율을 생각해도 된다. 즉 이를 통해 1년 동안의 return이 얼마인가를 알고 있어도 각 subinterval에 정의되는 이자율을 적용할 수 있는 것이다.

If r (without the "hat") denotes one plus the rate of interest over a *fixed* unit of calendar time, then over elapsed time t, r^t is the total return.[10] Observe that this measure of total return does not depend on n. As we have argued, we want to choose the dependence of $\hat{r}$ on n, so that

- **r, r^의 관계**

APR/EAR의 공식에 따라 아래와 같은 관계가 성립한다.

$$\hat{r}^n = r^t$$

$(r\wedge)^n = r^t$

for any choice of n. Therefore, $\hat{r} = r^{t/n}$. This last equation shows how $\hat{r}$ must depend on n for the total return over elapsed time t to be independent of n.

따라서 $r\wedge = r^{t/n}$ 이 된다.

We also need to define u and d in terms of n. At this point, there are two significantly different paths we can take. Depending on the definitions we choose, as $n \to \infty$ (or, equivalently, as $h \to 0$), we can have either a continuous or a jump stochastic process. In the first situation, very

[10] The scale of this unit (perhaps a day, or a year) is unimportant as long as r and t are expressed in the same scale.

small random changes in the stock price will be occurring in each very small time interval. The stock price will fluctuate incessantly, but its path can be drawn without lifting pen from paper. In contrast, in the second case, the stock price will usually move in a smooth deterministic way, but will occasionally experience sudden discontinuous changes. Both can be derived from our binomial process simply by choosing how u and d depend on n. We examine in detail only the continuous process that leads to the option pricing formula originally derived by Fischer Black and Myron Scholes. Subsequently, we indicate how to develop the jump process formula originally derived by John Cox and Stephen Ross.

Recall that we supposed that over each period the stock price would experience a one plus rate of return of u with probability q and d with probability $1 - q$. It will be easier and clearer to work instead with the natural logarithm of the one plus rate of return, $\log u$ or $\log d$. This

- 요약하면, **u가 될 가능성은 q, d가 될 가능성은 (1-q)**이다.

그런데 log를 취해서 계산하면 보다 계산이 수월해지기 때문에 log를 씌워서 **log(u), log(d)**를 쓰

Recall that we supposed that over each period the stock price would experience a one plus rate of return of u with probability q and d with probability $1-q$. It will be easier and clearer to work, instead, with the natural logarithm of the one plus rate of return, $\log u$ or $\log d$. This gives the continuously compounded rate of return on the stock over each period. It is a random variable which, in each period, will be equal to $\log u$ with probability q and $\log d$ with probability $1-q$.

Consider a typical sequence of five moves, say u, d, u, u, d. Then the final stock price will be $S^* = uduudS$; $S^*/S = u^3d^2$, and $\log(S^*/S) = 3\log u + 2\log d$. More generally, over n periods,

$$\log(S^*/S) = j\log u + (n-j)\log d = j\log(u/d) + n\log d$$

where j is the (random) number of upward moves occurring during the n periods to expiration. Therefore, the expected value of $\log(S^*/S)$ is

$$E[\log(S^*/S)] = \log(u/d) \bullet E(j) + n\log d$$

and its variance is

$$Var[\log(S^*/S)] = [\log(u/d)]^2 \bullet Var(j)$$

Each of the n possible upward moves has probability q. Thus, $E(j) = nq$. Also since the variance each period is $q(1-q)^2 + (1-q)(0-q)^2 = q(1-q)$, then $Var(j) = nq(1-q)$. Combining all of this, we have

$$E[\log(S^*/S)] = [q\log(u/d) + \log d]n \equiv \hat{\mu}n$$
$$Var[\log(S^*/S)] = q(1-q)[\log(u/d)]^2 n \equiv \hat{\sigma}^2 n$$

Let us go back to our discussion. We were considering dividing up our original longer time period (a day) into many shorter periods (a minute or even less). Our procedure calls for, over fixed length of calendar time t, making n larger and larger. Now if we held everything else constant while we let n become large, we would be faced with the problem we talked about earlier. In fact, we would certainly not reach a reasonable conclusion if either $\hat{\mu}n$ or $\hat{\sigma}^2 n$ went to zero or infinity as n became large. Since t is a fixed length of time, in searching for a realistic result, we must make the appropriate adjustments in u, d, and q. In doing that, we would at least want the mean and variance of the continuously compounded rate of return of the assumed stock price movement to coincide with that of the actual stock price as $n \to \infty$.

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- 요약하면, **u가 될 가능성은 q, d가 될 가능성은 (1-q)**이다.
  그런데 log를 취해서 계산하면 보다 계산이 수월해지기 때문에 log를 씌워서 **log(u), log(d)**를 쓰도록 한다.

- u, d, u, u, d를 가지고 generalization을 시켜보자. 이걸 어떻게 표현할까?
  만기시점 주가 S* 는, 제시된대로 5번을 던져서 uduud로 나왔다고 할 때 **S*=uduudS**로 볼 수 있다. 이 때 양변에 로그를 취해서 로그 수익률을 구해보면 아래와 같다.
  - **ln(S*/S) = 3ln(u) + 2ln(d)**

  이런걸 로그 수익율이라고 부른다.

- **일반화**
  이때 총 n번을 던져서 up이 j번 나왔다고 할 때, 이를 일반화시키면 아래와 같다.
  $$\ln(S^*/S) = j\ln(u) + (n-j)\ln(d) = j(\ln(u) - \ln(d)) + n\ln(d)$$

- **기대값(평균), 분산 구하기**
  기대값/분산을 구할 때 유의할 것은 j 만이 random variable이라는 점이다. 따라서 왼쪽과 같이 exp, var 식을 유도할 수 있다.

  [참고식]
  y=ax+b 일 때
  E(y) = a*E(x)+b
  V(y) = a^2V(x)

  그럼 식의 중간에 있는 E(j)와 V(j)를 없애주어야 하는데, **E(j)=nq, V(j)=nq(1-q)**가 된다. (확률분산의 기본적 정의에서 가져옴) 이를 대입 정리하면 왼쪽과 같은 식이 나오고, 이를 정리하면 새로운 변수 μ^과 σ^을 정의할 수 있다.

  그런데 문제는 우리는 n→∞이 될 경우를 조사하고자 하는 것인데, μ^n과 σ^n에는 각기 n이 붙어 있기 때문에 이는 n→∞으로 갈 때 0이나 무한대로 가 버리게 된다. 따라서 u, d, q를 적절하게 조정해 줄 필요가 있다.

- **Notation μ^, σ^**
  $$\hat{\mu} = q\log(u/d) + \log d$$
  $$\hat{\sigma}^2 = q(1-q)[\log(u/d)]^2$$

- **[중요중요] 이 부분부터가 시험문제이다**
  [참고] 이는 실제 숫자가 아닌데, 이에 대응하는 실세계의 값들이 바로 오른쪽의 값들이다. 즉 오른

Suppose we label the actual empirical values of $\hat{\mu}n$ and $\hat{\sigma}^2 n$ as μ and $\sigma^2 t$, respectively. Then we would want to choose u, d, and q so that

$$\underbrace{[q\log(u/d)+\log d]n}_{=\text{u\^{}nt}} \to \mu t \qquad \text{as } n \to \infty$$

$$\underbrace{q(1-q)[\log(u/d)]^2 n}_{=\sigma\text{\^{}}^2\text{n}} \to \sigma^2 t$$

A little algebra shows we can accomplish this by letting

$$u = e^{\sigma\sqrt{t/n}}, \quad d = e^{-\sigma\sqrt{t/n}}, \quad q = \frac{1}{2}+\frac{1}{2}(\mu/\sigma)\sqrt{t/n}$$

적지 않은 노력을 통해 u, d, q를 이처럼 정의할 수 있다. 이 증명은 추후 배우도록 한다.

In this case, for any n,

$$\hat{\mu}n = \mu t \quad \text{and} \quad \hat{\sigma}^2 n = [\sigma^2 - \mu^2(t/n)]\,t$$

Clearly, as $n \to \infty$, $\hat{\sigma}^2 n \to \sigma^2 t$ while $\hat{\mu}n = \mu t$ for all values of n.

Alternatively, we could have chosen u, d, and q so that the mean and variance of the future stock price for the discrete binomial process approach the prespecified mean and variance of the actual stock price as $n \to \infty$. However, just as we would expect, the same values will accomplish this as well. Since this would not change our conclusions, and it is computationally more convenient to work with the continuously compounded rates of return, we will proceed in that way.

This satisfies our initial requirement that the limiting means and variances coincide, but we still need to verify that we are arriving at a sensible limiting probability distribution of the continuously compounded rate of return. The mean and variance only describe certain aspects of that distribution.

For our model, the random continuously compounded rate of return over a period of length t is the sum of n independent random variables, each of which can take the value $\log u$ with probability q and $\log d$ with probably $1-q$. We wish to know about the distribution of this sum as n becomes large and q, u, and d are chosen in the way described. We need to remember that as we change n, we are not simply adding one more random variable to the previous sum, but instead are changing the probabilities and possible outcomes for every member of the sum. At this point, we can rely on a form of the central limit theorem which, when applied to our problem, says that, as $n \to \infty$, if

$$\frac{q|\log u - \hat{\mu}|^3 + (1-q)|\log d - \hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}} \to 0$$

then

$$\text{Prob}\left[\left(\frac{\log(S^*/S) - \hat{\mu}n}{\hat{\sigma}\sqrt{n}}\right) \le z\right] \to N(z)$$

- **[중요중요] 이 부분부터가 시험문제이다**

  [참고] 이는 실제 숫자가 아닌데, 이에 대응하는 실세계의 값들이 바로 오른쪽의 값들이다. 즉 오른쪽은 "진짜 숫자"이며, 특히 만기를 의미하는 숫자이다. 만기가 10년이라고 하면, 매 평균적으로 10%씩 올라가며 변동성은 시간이 가면 갈수록 커진다. 그래서 t라고 하는 부분을 포함하고 있는 것이다. 우리가 원하는 것은 n을 무한대로 만드는 것이다. 따라서 이렇게 u와 d를 정의한다.

  이를 통해 궁극적으로는 n→∞으로 갈 때 μ^nt → μt 로 가고, σ^2n → σ²t 로 감을 보이는 것이다.

- **u, d, q를 검증하기**

  u, d, q를 뽑아내는 것은 쉽지 않으므로, 아래와 같이 대입해서 맞는지를 검증해 보자.

$$[q\ln(u/d)+\ln d]n$$
$$=\left[[\frac{1}{2}+\frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{t}{n}}]2\sigma\sqrt{t/n}-\sigma\sqrt{t/n}\right]n$$
$$=\left[\sigma\sqrt{(t/n)}+\mu\frac{t}{n}-\sigma\sqrt{\frac{t}{n}}\right]n=\mu t$$

따라서 만족함을 알 수 있다.

또한,

$$q(1-q)[\ln(u/d)]^2 n$$
$$=(\frac{1}{2}+\frac{1}{2}(\frac{\mu}{\sigma})\sqrt{(t/n)})(\frac{1}{2}-\frac{1}{2}(\frac{\mu}{\sigma})\sqrt{(t/n)})(2\sigma\sqrt{(t/n)})^2 n$$
$$=(\frac{1}{4}-\frac{1}{4}\frac{\mu^2}{s^2}\frac{t}{n})4\sigma^2\frac{t}{n}n=\sigma^2 t-\mu^2\frac{t^2}{n}=>\sigma^2 t$$

따라서 u, d, q가 n→∞ 일 때 주어진 조건을 만족함을 확인할 수 있다.

여하튼 이것은 굉장히 일반적인 형태의 Form이다. 이 때 u=exp(어쩌구..) 형태의 함수는 cancellation 효과 때문에 많이 쓰인다.

[참고] 그런데 이것이 unique solution이 아니다. Q=1/2 인 다른 식으로 만들어버려도 만족하는 식이 있다. 즉 이것만이 유일한 solution은 아니라는 것이다.

- **이 논문에서 보이고자 하는 최종 목표**

이건 중심 극한 정리이다. 우리가 하려는 최종 목적은 Binomial option pricing에서 아래 조건이 성립함을 보이는 것이다.

$$S\Phi(a;n,p) - Kr^{-n}\Phi(a;n,p)$$

이것이 아래와 같이 수렴함을 보이는 것이다.

$$SN(x) - Kr^{-t}N(x - s\sqrt{t})$$

즉 숫자를 막 집어넣었을 때 N으로 가는지를 보이는 것의 우리의 목표인 것이다. 그리고 이 중에서 시험문제가 나가게 된다. (젠장)

where $N(z)$ is the standard normal distribution function. Putting this into words, as the number of periods into which the fixed length of time to expiration is divided approaches infinity, the probability that the standardized continuously compounded rate of return of the stock through the expiration date is not greater than the number z approaches the probability under a standard normal distribution.

The initial condition says roughly that higher-order properties of the distribution, such as how it is skewed, become less and less important, relative to its standard deviation, as $n \to \infty$. We can verify that the condition is satisfied by making the appropriate substitutions and finding

$$\frac{q|\log u - \hat{\mu}|^3 + (1-q)|\log d - \hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}} = \frac{(1-q)^2 + q^2}{\sqrt{nq(1-q)}}$$

- 이 식의 구체적인 증명 방법은 다음 페이지에 나와 있다. (조교 연습 시간에도 풀었음)

which goes to zero as $n \to \infty$ since $q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{t/n}$. Thus, the multiplicative binomial model for stock prices includes the lognormal distributionas a limiting case.

Black and Scholes began directly with continuous trading and the assumption of a lognormal distribution for stock prices. Their approach relied on some quite advanced mathematics. However, since our approach contains continuous trading and the lognormal distribution as a limiting case, the two resulting formulas should then coincide. We will see shortly that this is indeed true, and we will have the advantage of using a much simpler method. It is important to remember, however, tha t the economic arguments we used to link the option value and the stock price are exactly the same as those advanced by Black and Scholes (1973) and Merton (1973, 1977).

The formula derived by Black and Scholes, rewritten in terms of our notation, is

Black-Scholes Option Pricing Formula

The formula derived by Black and Scholes, rewritten in terms of our notation, is

Black-Scholes Option Pricing Formula

$$C = SN(x) - Kr^{-t}N(x - \sigma\sqrt{t})$$

where

$$x \equiv \frac{\log(S / Kr^{-t})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

: 이게 initial condition이다.

We now wish to confirm that our binomial formula converges to the Black-Scholes formula when t is divided into more and more subintervals, and $\hat{r}$, u, d, and q are chosen in the way we described —that is, in a way such that the multiplicative binomial probability distribution of stock prices goes to the lognormal distribution.

- 이제 여기서부터 binomial formula가 t가 계속 쪼개질 때 BSM으로 수렴함을 보이게 된다.

For easy reference, let us recall our binomial option pricing formula:

$$C = S\phi[a;n,p'] - K\hat{r}^{-n}\phi[a;n,p]$$

The similarities are readily apparent. $\hat{r}^{-n}$ is, of course, always equal to r^{-t}. Therefore, to show the two formulas converge, we need only show that as $n \to \infty$

$$\boxed{\phi[a;n,p'] \to N(x) \quad \text{and} \quad \phi[a;n,p] \to N(x - \sigma\sqrt{t})}$$

We will consider only $\phi[a; n, p]$, since the argument is exactly the same for $\phi[a; n, p']$.

p'

The complementary binomial distribution function $\phi[a; n, p]$ is the probability that the sum of n random variables, each of which can take on the value 1 with the probability p and 0 with the probability $1 - p$, will be greater than or equal to a. We know that the random value of this sum, j, has mean np and standard deviation $\sqrt{np(1-p)}$. Therefore,

$$1 - \phi[a, n, p] = \text{Prob}[j \le a - 1] = \text{Prob}\left[\frac{j - np}{\sqrt{np(1-p)}} \le \frac{a - 1 - np}{\sqrt{np(1-p)}}\right]$$

- **Step 1**
  C= S 어쩌구 + K 어쩌구의 Binomial 계산식에서부터 시작한다. 이는 discrete 버전이므로, n→∞로 갈 때 Φ[a;n,p] 함수가 왼쪽과 같이 정규 분포 함수로 수렴함을 보이면 된다.

  우선 Φ[a;n,p] 함수가 p의 확률로 1을 가지고 (1-p)의 확률로 0을 가지는데 그 값이 a보다 클 확률임을 알고 있다. 이 경우 random variable은 j가 되며, 이의 평균은 np, 표준편차는 √{np(1-p)} 가 된다. 따라서 1-Φ[a;n,p] 는 j가 a-1보다 적게 나온다는 의미와 같다. **여기에서 a-1을 해 주는 이유는 continuous한 버전이 아니라 discrete한 버전이기 때문이다.** 여하튼 정리하면 왼쪽과 같은 식을 얻을 수 있다.

- **Step 2**

  Measure는 아래와 같다.

$$p = \frac{\hat{r} - d}{u - d}$$

j, has mean np and standard deviation $\sqrt{np(1-p)}$. Therefore,

$$1-\phi[a,n,p]=\text{Prob}[j \le a-1] = \text{Prob}\left[\frac{j-np}{\sqrt{np(1-p)}} \le \frac{a-1-np}{\sqrt{np(1-p)}}\right]$$

Now we can make an analogy with our earlier discussion. If we consider a stock which in each period will move to uS with probability p and dS with probability $1-p$, then $\log(S^*/S)=j \log(u/d)+n\log d$. The mean and variance of the continuously compounded rate of return of this stock are

$$\hat{\mu}_p = p\log(u/d)+\log d \quad \text{and} \quad \hat{\sigma}_p^2 = p(1-p)[\log(u/d)]^2$$

Using these equalities, we find that

$$\frac{j-np}{\sqrt{np(1-p)}} = \frac{\log(S^*/S)-\hat{\mu}_p n}{\hat{\sigma}_p\sqrt{n}}$$

Recall from the binomial formula that

$$a-1=\log(K/Sd^n)/\log(u/d)-\varepsilon = [\log(K/S)-n\log d]/\log(u/d)-\varepsilon,$$

where ε is a number between zero and one. Using this and the definitions of $\hat{\mu}_p$ and $\hat{\sigma}_p^2$, with a little algebra, we have

$$\frac{a-1-np}{\sqrt{np(1-p)}} = \frac{\log(K/S)-\hat{\mu}_p n-\varepsilon\log(u/d)}{\hat{\sigma}_p\sqrt{n}}$$

Putting these results together,

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Measure는 아래와 같다.

$$p=\frac{\hat{r}-d}{u-d}$$

$$q=\frac{1}{2}+\frac{1}{2}(\mu/\sigma)\sqrt{t/n}$$

이제 준비가 되었으므로 Prob 공식을 분해해보자. 아래 계수들을 대입하면 된다.

- **Step 3**

Continuous 버전에서 구한 아래의 j에 대한 식을

$$\ln(S^*/S)=j(\ln(u)-\ln(d))+n\ln(d)$$

정리하면 아래와 같이 나온다.

$$j=\frac{\ln(S^*/S)-n\ln d}{\ln(u/d)}$$

마찬가지로 아래의 continuous 버전의 식들을 정리하면

$$\hat{\mu}_p = p\log(u/d)+\log d$$

$$\hat{\sigma}_p^2 = p(1-p)[\log(u/d)]^2$$

아래와 같은 답을 얻을 수 있다.

$$np=\frac{\hat{\mu}n-n\ln d}{\ln(u/d)}$$

$$\sqrt{np(1-p)}=\frac{\hat{\sigma}\sqrt{n}}{\ln(u/d)}$$

이걸 대입해서 정리하면 오른쪽의 식이 나오는 것이다.

- **Step 4**

이제 Prob 함수의 우변, 즉 a-1에 대해서 풀어 주어야 한다. 10page에 있는 a의 정의에서 뽑아오면

$$u^a d^{n-a} S - K \ge 0$$

의 식이다. 이를 log를 씌워서 정리하면,

$$a\ln u+(n-a)\ln d+\ln S \geqq \ln K$$

$$a \geqq \frac{\ln(K/S)-n\ln d}{\ln(u/d)}$$

가 된다. 그런데 a-1을 정의해야 하므로, 계수 ε 를 붙여서 아래와 같이 정리한다.

$$a-1=\frac{\ln(K/S)-n\ln d}{\ln(u/d)}-\epsilon$$

이를 Prob[ ]의 우변에 대입해서 정리하면 오른쪽의 식이 나오는 것이다.

$$1 - \phi[a; n, p] = \text{Prob}\left[\frac{\log(S^*/S) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}} \le \frac{\log(K/S) - \hat{\mu}_p n - \varepsilon \log(u/d)}{\hat{\sigma}_p \sqrt{n}}\right]$$

We are now in a position to apply the central limit theorem. First, we must check if the initial condition,

$$\frac{p\left|\log u - \hat{\mu}_p\right|^3 + (1-p)\left|\log d - \hat{\mu}_p\right|^3}{\hat{\sigma}_p \sqrt{n}} = \frac{(1-p)^2 + p^2}{\sqrt{np(1-p)}} \to 0$$

as $n \to \infty$, is satisfied. By first recalling that $p \equiv (\hat{r} - d)/(u - d)$, and then $\hat{r} = r^{t/n}$, $u = e^{\sigma\sqrt{t/n}}$, and $d = e^{-\sigma\sqrt{t/n}}$, it is possible to show that as $n \to \infty$,

$$p \to \frac{1}{2} + \frac{1}{2}\left(\frac{\log r - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\frac{t}{n}}$$

As a result, the initial condition holds, and we are justified in applying the central limit theorem.

To do so, we need only evaluate $\hat{\mu}_p n$, $\hat{\sigma}_p^2 n$ and $\log(u/d)$ as $n \to \infty$.[11] Examination of our discussion for parameterizing q shows that as $n \to \infty$

---

[11] A surprising feature of this evaluation is that although $p \ne q$ and thus $\hat{\mu}_p \ne \hat{\mu}$ and $\hat{\sigma}_p \ne \hat{\sigma}$, nonetheless $\hat{\sigma}_p \sqrt{n}$ and $\hat{\sigma}\sqrt{n}$ have the same limiting value as $n \to \infty$. By contrast, since $\mu \ne \log r - \left(\frac{1}{2}\sigma^2\right)$, $\hat{\mu}_p n$ and $\hat{\mu} n$ do not. This results from the way we needed to specify u and d to obtain convergence to a lognormal distribution. Rewriting this as $\sigma\sqrt{t} = (\log u)\sqrt{n}$, it is clear that the limiting value σ of the standard deviation does not depend on p or q, and hence must be the same for either. However, at any point before the limit, since

$$\hat{\sigma}^2 n = \left(\sigma^2 - \mu^2 \frac{t}{n}\right)t \quad \text{and} \quad \hat{\sigma}_p^2 n = \left[\sigma^2 - \left(\log r - \frac{1}{2}\sigma^2\right)^2 \frac{t}{n}\right]t$$

$\hat{\sigma}$ and $\hat{\sigma}_p$ will generally have different values.

The fact that $\hat{\mu}_p n \to \left(\log r - \frac{1}{2}\sigma^2\right)t$ can also be derived from the property of the lognormal distribution that

$$\log E[S^*/S] = \mu_p t + \frac{1}{2}\sigma^2 t$$

where E and μ_p are measured with respect to probability p. Since $p = (\hat{r} - d)/(u - d)$, it follows that $\hat{r} = pu + (1-p)d$. For independently distributed random variables, the expectation of a product equals the product

- Step 5

이제 중심 극한 정리를 이용하는 핵심적인 부분이 남았다.

아래의 공식

$$p = \frac{\hat{r} - d}{u - d}$$

에서 n→∞ 일 때를 구하여 다음으로 수렴함을 보이면 된다.

$$q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{t/n}$$

면 된다. 다른 말로 이는 discrete version을 continuous version으로 바꾼다는 의미와 동일하다. 아래 변수들을 각기 치환하면 된다.

$$u = e^{\sigma\sqrt{t/n}}$$

$$d = e^{-\sigma\sqrt{t/n}}$$

$$\hat{r} = r^{t/n}$$

그런데 그냥은 풀리지 않으므로 아래와 같이 Taylor 전개를 이용한다.

$$f(x) - f(0) = f'(0)(x - 0) + \frac{f'(0)}{2!}(x - 0)^2 + \dots$$

u와 d를 이를 이용하여 전개하면

$$u = e^{\sigma\sqrt{t/n}} = e^{\sigma x} = 1 + \sigma e^{\sigma x}|_{x=0} + \frac{1}{2}\sigma^2 e^{\sigma x}|_{x=0} x^2 + \dots$$

$$d = e^{-\sigma\sqrt{t/n}} = e^{-\sigma x} = 1 - \sigma e^{\sigma x}|_{x=0} + \frac{1}{2}\sigma^2 e^{\sigma x}|_{x=0} x^2 - \dots$$

이 때

$$x = \sqrt{t/n}$$

따라서 u와 d는 아래와 같다

$$\log E[S^*/S] = \mu_p t + \frac{1}{2}\sigma^2 t$$

where E and μ_p are measured with respect to probability p. Since $p = (\hat{r} - d)/(u - d)$, it follows that $\hat{r} = pu + (1-p)d$. For independently distributed random variables, the expectation of a product equals the product of their expectations. Therefore,

$$E[S^*/S] = [pu + (1-p)d]^n = \hat{r}^n = r^t$$

Substituting r^t for $E[S^*/S]$ in the previous equation, we have

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$$\hat{\mu}_p n \to \left(\log r - \frac{1}{2}\sigma^2\right) t \quad \text{and} \quad \hat{\sigma}_p \sqrt{n} \to \sigma\sqrt{t}$$

Furthermore, $\log(u/d) \to 0$ as $n \to \infty$.

For this application of the central limit theorem, then, since

$$\frac{\log(K/S) - \hat{\mu}_p n - \varepsilon \log(u/d)}{\hat{\sigma}_p \sqrt{n}} \to z = \frac{\log(K/S) - \left(\log r - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}$$

we have

이 때

$$x = \sqrt{t/n}$$

따라서, u와 d는 아래와 같다.

$$u = 1 + \sigma x + \frac{1}{2}\sigma^2 x^2$$

$$d = 1 + \sigma x - \frac{1}{2}\sigma^2 x^2$$

r^ 을 전개할 때에는 아래의 공식을 이용해서 전개한다.

$$\frac{da^{u(x)}}{dx} = \ln a \cdot a^{u(x)} \frac{du(x)}{dx}$$

그럼 r^은 아래와 같이 전개되고,

$$\hat{r} = r^{t/n} = r^{x^2} = 1 + (\ln r \cdot r^{x^2} \cdot 2x)|_{x=0} x + \frac{1}{2}(\ln r \cdot r^{x^2} \cdot 2 + (2x\ln r)^2 r^{x^2})|_{x=0} \cdots$$

$$\hat{r} = 1 + \ln r x^2$$

와 같이 된다. 이를 p 식에 대입하면,

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\ln r - \frac{1}{2}\sigma^2}{\sigma}\right)x = \frac{1}{2} + \frac{1}{2}\left(\frac{\ln r - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\frac{t}{n}}$$

이는 initial condition을 만족함을 알 수 있다.

따라서, p라는 measure를 사용할 때

$$\hat{\mu} n \to \mu t$$

$$\hat{\mu}_p n \to (\ln r - \frac{1}{2}\sigma^2)t$$

를 만족함을 알 수 있다.

- **Step 6.**

  이제 이를 이용하여 RHS를 discrete 버전에서 continuous 버전으로 바꾸어 보자. 그러면,

$$\frac{\ln(K/S) - \hat{\mu}_p n - \epsilon \ln(u/d)}{\hat{\sigma}_p \sqrt{n}}$$

$$= \frac{\ln(K/S) - (\ln r - \frac{1}{2}\sigma^2)t}{\hat{\sigma}_p \sqrt{n}}$$

$$= \frac{\ln(K/S) - (\ln r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

$$1-\phi[a;n,p] \rightarrow N(z) = N\left[\frac{\log(Kr^{-t}/S)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right]$$

이 때 ln(u/d)는 n->∞ 로 가면서 사라지게 된다.

그러면 최종적으로 주어진 식을 만족하게 된다.

The final step in the argument is to use the symmetry property of the standard normal deviation distribution that $1 - N(z) = N(-z)$. Therefore, as $n \rightarrow \infty$

$$\phi[a;n,p] \rightarrow N(-z) = N\left[\frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right] = N(x - \sigma\sqrt{t})$$

Since a similar argument holds for $\phi[a;\ n,\ p\ ]$, this completes our demonstration that the binomial option pricing formula contains the Black-Scholes formula as a limiting case.[12,13]

$$\mu_p = \log r - \frac{1}{2}\sigma^2$$

[12] The only difference is that, as $n \rightarrow \infty$, $p' \rightarrow \frac{1}{2} + \frac{1}{2}\left[\left(\log r + \frac{1}{2}\sigma^2\right)/\sigma\right]\sqrt{t/n}$. Further, it can be shown that as $n \rightarrow \infty$, $\Delta \rightarrow N(x)$. Therefore, for the Black-Scholes model, $\Delta S = SN(x)$ and $B = -Kr^{-t}N(x - \sigma\sqrt{t})$.

[13] In our original development, we obtained the following equation (somewhat rewritten) relating the call prices in successive periods:

$$\left(\frac{\hat{r}-d}{u-d}\right)C_u + \left(\frac{u-\hat{r}}{u-d}\right)C_d - \hat{r}C = 0$$

By their more difficult methods, Black and Scholes obtained directly a partial differential equation analogous to our discrete-time difference equation. Their equation is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (\log r)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - (\log r)C = 0.$$

The value of the call, C, was then derived by solving this equation subject to the boundary condition $C^* = \max[0, S^* - K]$.

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As we have remarked, the seeds of both the Black-Scholes formula and a continuous-time jump process formula are both contained within the binomial formulation. At which end point we

As we have remarked, the seeds of both the Black-Scholes formula and a continuous-time jump process formula are both contained within the binomial formulation. At which end point we arrive depends on how we take limits. Suppose, in place of our former correspondence for u, d, and q, we instead set

$$u = u, \quad d = e^{\zeta(t/n)}, \quad q = \lambda(t/n).$$

This correspondence captures the essence of a pure jump process in which each successive stock price is almost always close to the previous price ($S \to dS$), but occasionally, with low but continuing probability, significantly different ($S \to uS$). Observe that, as $n \to \infty$, the probability of a change by d becomes larger and larger, while the probability of a change by u approaches zero.

With these specifications, the initial condition of the central limit theorem we used is no longer satisfied, and it can be shown the stock price movements converge to a log-Poisson rather than a lognormal distribution as $n \to \infty$. Let us define

$$\Psi[x; y] \equiv \sum_{i=x}^{\infty} \frac{e^{-y} y^i}{i!}$$

as the complementary Poisson distribution function. The limiting option pricing formula for the above specifications of u, d and q is then

Jump Process Option Pricing Formula

$$\boxed{\begin{gathered} C = S\Psi[x; y] - Kr^{-t}\Psi[x; y/u], \\ \text{where} \\ y \equiv (\log r - \zeta)ut/(u-1), \\ \text{and} \\ x \equiv \text{the smallest non-negative integer} \\ \text{greater than } (\log(K/S) - \zeta t)/\log u. \end{gathered}}$$

A very similar formula holds if we let $u = e^{\zeta(t/n)}$, $d = d$, and $1 - q = \lambda(t/n)$.

Based on our previous analysis, we would now suspect that, as $n \to \infty$, our difference equation would approach the Black-Scholes partial differential equation. This can be confirmed by substituting our definitions of $\hat{r}$, u, d in terms of n in the way described earlier, expanding C_u, C_d in a Taylor series around $(e^{\sigma\sqrt{h}}S, t-h)$ and $(e^{-\sigma\sqrt{h}}S, t-h)$, respectively, and then expanding $e^{\sigma\sqrt{h}}$, $e^{-\sigma\sqrt{h}}$, and r^h in a Taylor series, substituting these in the equation and collecting terms. If we then divide by h and let $h \to 0$, all terms of higher order than h go to zero. This yields the Black-Scholes equation.